Advanced Polynomial Cancellation Coded OFDM

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Abstract

The conventional Polynomial cancellation coded orthogonal frequency division multiplexing (PCC-OFDM) uses a linear combination of several OFDM single carriers. The frequency spectrum of the combination is decaying faster than the frequency spectrum of a single carrier. The sidelobes of one carrier is partly cancelled by the others. This is also called the ICI self cancellation property.

We introduce here a new method to construct a hole orthonormal basis on top of several OFDM single carriers. The Spectrum of each vector in the linear subspace spanned by this basis is decaying faster than a OFDM single carrier.

On base of this new method, we could define new modulation schemes, which are decaying fast in frequency domain and are very spectral efficient.

1 Introduction

The frequency domain representation of a timelimited function between the points $-\pi$ and $+\pi$ is given by (see Appendix A):

$$\mathcal{F}(\omega) = \sum_{k=-\infty}^{\infty} c_k \cdot \frac{\sin(\omega\pi - k\pi)}{\omega\pi - k\pi} \tag{1}$$

The translations of the functions $\operatorname{sinc}(x) := \sin(\pi x)/\pi x$ are a orthonormal basis of the frequency domain (Hilbert space) of all timelimited functions between $-\pi$ and $+\pi$.

We intend to construct fast decaying modulation functions. It is suffice for us to use only a finite sum of translated sinc functions. For example we are looking at a calculation with only 3 terms.

$$\begin{aligned}
\mathcal{G}(x) &= c_1 \cdot \frac{\sin(x+\pi)}{x+\pi} + c_2 \cdot \frac{\sin(x+2\pi)}{x+2\pi} + c_3 \cdot \frac{\sin(x+3\pi)}{x+3\pi} && (2)\\
&= \left(\frac{-c_1}{x+\pi} + \frac{c_2}{x+2\pi} + \frac{-c_3}{x+3\pi}\right) \cdot \sin(x) && (3)\\
&= \left(\frac{-c_1(x+2\pi)(x+3\pi) + c_2(x+\pi)(x+3\pi) - c_3(x+\pi)(x+2\pi)}{(x+\pi)\cdot(x+2\pi)\cdot(x+3\pi)}\right) \cdot \sin(x) && (4)\\
&= \left(\frac{(-c_1+c_2-c_3)\cdot x^2 + (-c_1\cdot 5\pi + c_2\cdot 4\pi - c_3\cdot 3\pi)\cdot x - c_1 6\pi^2 + c_2 3\pi^2 - c_3 2\pi^2}{(x+\pi)\cdot(x+2\pi)\cdot(x+3\pi)}\right) \cdot \sin(x) && (5)
\end{aligned}$$

At the numerator is a polynomial of x with coefficients composed of the c_k. If the polynomial in the numerator is of low degree, the function is decaying fast. If the original coefficients c_k are in special linear relations, you can even make the numerator to a constant. For example we are choosing $c_1 = \frac{1}{2}\tilde{c}$, $c_2 = \tilde{c}$, $c_3 = \frac{1}{2}\tilde{c}$. This gives us following result:

$$\mathcal{G}(x) = \left(\frac{-\pi^2 \cdot \tilde{c}}{(x+\pi)\cdot(x+2\pi)\cdot(x+3\pi)}\right) \cdot \sin(x) \tag{6}$$

If the original coefficients c_k are in special linear relations, the sidelobes of the single carriers are self cancelling each other. Amplitudes of standard OFDM single carriers (without cyclic prefix and windowing) are decaying with $\mathcal{O}(\frac{1}{x})$. Amplitudes of a linear combination of n single carriers can decay in the best case with $\mathcal{O}(\frac{1}{x^n})$.

This possibility of self cancelation of the sidelobes is already discussed in literature. Zhao, Y. and Haggman, S.G. has called this property ICI self-cancellation [1, 2].

Armstrong et al. [3–5] has called the resulting modulation scheme Polynomial Cancellation Coding (PCC-OFDM).

This type of modulation is given, if you modulate the fast decaying linear combinations, instead of the single carriers. This modulation is generating less adjacent channel power, has less Inter-Channel-Interference (ICI) and is less sensitive to frequency offsets.

With the loss of modulateable carriers you lose usally spectral efficiency. If you have less modulateable carriers, you can have a higher signal to noise ratio on the remaining carriers. But this is usally not as good as having more orthogonal carriers.

There are other methods trying to exploit the self-cancellation property. Subcarrier weighting [6], cancellation carriers [7] or multiple-choice sequences [8] are such methods.

2 Construction of new types of modulation

In PCC-OFDM you have choosen only one linear combination out of n carriers. We want to choose several linear combinations forming a orthonormal basis. For modulation we modulate each new basis vector distinct from each other. So we can reduce the loss of degrees of freedom and remain the spectral efficiency.

This was the intermediate result from above:

The combined frequency domain function of n OFDM single carriers you can write as:

$$\mathcal{G}(x) \quad = \quad \frac{Polynomial_{n-1}(x)}{Polynomial_{n}(x)} \cdot \sin(x) \tag{7}$$

The denominator is a fix polynomial of degree n. The denominator is not dependent on the coefficients c_k of the OFDM single carriers. At the roots of the denominator are at the roots of the Sinus. Thus the term is well defined.

In the numerator can be dependent on the coefficients c_k a polynomial of degree $n-1$. If you want to have a high decay of hole term against $+/-\infty$, there should be a polynomial of low order in the numerator.

I want also to mention here, that we have a nice chain of Isomorphism and Homomorphism. The Hilbert space of time limited functions with Norm $\mathcal{L}^2$ has a Isomorphism to the fourier space with infinit numbers of coefficients and the euclidean norm $\|\cdot\|_2$. Since we could only use a finite number of carriers for practical reasons, we have a Homomorphism to the subspace of n carriers ($\mathbb{R}^n$) with the euclidean norm $\|\cdot\|_2$. And further we have now a Isomorphism into the space of polynomial functions of degree $n-1$ in the numerator. The norm of this polynomial space is induced by Isomorphism.

How could we calculate the norm of a polynomial in this induced norm? We recalculate the coefficients of the OFDM single carriers and using the norm on top of this coefficients.

This coefficeints are given from the evaluation of the polynomial at the position of the OFDM single carrier and a devision through a fix weighting. This weighting is given by the Sinus and the polynomial in the denominator. The Singularity of the Sinus is lifting the root of the polynomial in the denominator. But don't forget the resulting sign at each second carrier.

After this necessary preparations, we could easily construct new modulation schemes.

All orthonormal basis vectors in the space of polynomials are defining a new modulation. On each basis vector you can independently apply a QAM or PSK-Modulation.

We want to construct such a basis of fast decaying basis vectors. We are using the linear subspace of all polynomials of degree $m-1$ with $m \leq n$. The decay of the amplitude in the frequency domain is then proportional to $\mathcal{O}(\frac{1}{x^{1+n-m}})$.

A basis of this linear subspace are for example the polynomials $x^0, x^1, ..., x^{m-2}, x^{m-1}$. With the Gram–Schmidt process you can easily made this to a orthonormal basis. For each basis vector of such a constructed basis, the amplitude in the frequency domain will fall with at least $\mathcal{O}(\frac{1}{x^{1+n-m}})$.

OFDM is resistent against multipath fading, because its subcarriers are localized in frequency domain. Our newly constructed orthonormal basis should also be localized in frequency domain and should not be dispersed to much. Our first approach is perhaps the opposite. Diffuse in the frequency domain, localised in the time domain.

Since you can rotate and reflect these basis vectors you can find a better basis for multipath fading conditions. Perhaps you can write a numerical search algorithm which finds this basis. A good criterion would be the sum (or max value) of variances of the basis vectors in the frequency domain. If you don't want to use the variance, you could also look at the max-values of the basis vectors with norm 1 in frequency domain.

Good seeds of a numerical search algorithm are given by Lagrange polynomials. A basis of Lagrange polynomials are given through m roots. Every basis polynomial is using $m-1$ roots and is usually set at the omitted root to 1. We can now try to guess good positions for roots and use the Gram–Schmidt process to make them orthonormal.

If we could find exactly the roots, so that the Lagrange polynomials are orthogonal in the induced norm, we would also have a good orthonormal basis for multipath fading conditions. The set of roots is given by m variables. The orthogonality between basis functions gives us $m-1$ equation. The last degree of freedom is used to center the basis functions in the intervall. This should make the solution unique, if it exists.

A better question is: Does such a solution exist? Perhaps not all Polynomials have $m-1$ roots. The roots are perhaps complex numbers instead of real numbers. In practice this roots exists and they are real.

To find such roots, I have developed a approximation procedure. This is based on the Newton's method and implies common roots. The optimisation criterion is the orthogonality of the basis vectors.

3 Computational demand

After we selected m polynomials, we can genarate a real matrix, which transforms n complex values of the OFDM single carriers into complex values in the m dimensional basis.

So we have a $m \times n$-Matrix. We need for a matrix multiplication $2 \cdot n \cdot m$ real multiplications or $\frac{1}{2} \cdot n \cdot m$ complex multiplications.

A such matrix-multiplication is for huge n and m computational expensive. If you have a lot of carriers, you should devide the carriers into groups and use the new method only on the groups. This reduces also Inter-Channel-Interference (ICI).

If you have N OFDM single carriers and a group size of n. So you have $\frac{N}{n}$ groups. The computational demand ist then $\frac{N}{n} \cdot 2 \cdot n \cdot m$ real multiplications.

Example:

For $N = 1024$, $n = 16$ and $m = 12$ you have $1024 \cdot 2 \cdot 12 = 24576$ real multiplications for all matrix multiplications. The also necessary Fourier transform needs additional $2 \cdot 1024 \cdot \log_2(1024) = 20480$ real multiplications (radix-2 algorithm). The amplitude in this example is decaying with $\mathcal{O}(\frac{1}{x^5})$, The power spectrum is decaying with $\mathcal{O}(\frac{1}{x^{10}})$.

There seems to be the chance of a fast version. A simple idee is to combine the matrix multiplication with the first $log_2(n)$ steps of the FFT. But this results in a complex matrix multiplication instead of a real matrix multiplication. So this would not help to reduce multiplications.

4 Application in practice

The new modulation was constructed for exactly one determined time period. You can not apply a cyclic prefix. But you can prefix the signal with zeros. (Zero-Padding or Zero-Prefix). In the receiver you can take the guard intervall and add it to the begin of the analyzed intervall. This reconstructs the orthogonality destroyed by multipath fading. The disadvantage is, that you copy additional noise to the signal.

The Guard-Intervall can be shorter, if you have dropped several polynomial degrees $(n-m)$. In this case the functions in time domain are at the interval endpoints $n-m-1$ times continuous differentiable (We assume zero-padding). The time domain functions at the endpoints are increasing or decreasing slow. In this case they are robust against multipath fading even without guard intervall.

OFDM has been often applied with cyclic prefix and windowing. Windowing has needed some additional time inside guard interval. Since we don't need windowing, we can adjust our guard interval.

To reduce adjacent channel power OFDM has left guard carriers unmodulated at the edges of the transmission spectrum (guard band). Since with the new modulation scheme the power spectrum is decreasing faster, you can use more carriers. At the edges of the transmission spectrum you can use more dropped dimensions $n - m$ as in the middle. So you can use the transmission spectrum more efficient.

The peak-to-average power ratio (PAPR) is bad. You should combine this modulation with a method to reduce PAPR. Most PAPR-Reduction methods used for OFDM are also applicable to the proposed modulation. There is an overview article of peak-to-average power ratio reduction techniques from Han, S.H. and Lee, J.H. [9].

An important difference to OFDM is the fact that the power envelope in time is not constant. The amplitudes are falling smooth to zero at the tails of the transmission symbol in time domain. This results in a somewhat worser PAPR. But you can easily combat this disadvantage by translating some carriers in time domain.

If you have a lot of carriers, and you have divided them into groups, you can translate hole groups in time domain, but you must preserve orthogonality.

If your group size is an exponent of 2 (2^n members), you can do the translation inside a classical FFT. The first n steps of the inverse FFT in the transmitter will only do multiplications and additions inside your groups. More precisely the first n steps will transform the coefficients of the carriers of your group to the time domain representation of your groups. The additional steps of the FFT will only do some frequency multiplexing of your groups.

So the ideal moment to do the translations and mixing with prevoious or next symbol data would be after n steps of the FFT. The orthogonality in this case would be preserved. The computational demand for translations is very low. No multiplications or additions would be required. Demand of computational ressources are more continous in time and perhaps you achieve a better latency.

	Advanced PCC-OFDM	OFDM with CP and windowing
Multipath fading	very good	very good
Spectral efficiency	can be even better, depends on parameters	good
Adjacent channel power	can be very good, depends on parameters	average, depends on carrier count
PAPR	bad	bad
ICI	very good between groups, bad inside groups	bad
Computational demand	higher, depends on groupsize	today, you can say low
Scalability	very good	good

Table 1: Feature comparison table, Note: No BER-Simulations have been done

5 TODO

Bit-Error-Rate - Simulations under various conditions. Comparision with other modulation methods.

Usage in practice.

If you not use all lost degrees for a fast decay in frequency domain, you can perhaps optimize to other criterions. (Stability against frequency offsets, frequency dispersion)

Fast version of calculation

6 Changes

Version 1.1 (April 3, 2010)

Added method for PAPR-Reduction through translation of carriers in time domain.

Version 1.0 (February 6, 2010)

Initial publication

7 Appendix A

Time limited Functions between $-\pi$ und $+\pi$ you can write as:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cdot \cos(kt) + b_k \cdot \sin(kt) \cdot 1_{[-\pi,\pi]}(t)) \tag{8}$$

We use the complex Version:

$$f(t) = \frac{1}{\sqrt{2\pi}} \cdot \sum_{k=-\infty}^{\infty} c_k \cdot \mathrm{e}^{\mathrm{i}kt} \cdot 1_{[-\pi,\pi]}(t) \tag{9}$$

In the frequency domain you retrieve:

$$\begin{aligned}
\mathcal{F}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \sum_{k=-\infty}^{\infty} c_k \cdot \mathrm{e}^{\mathrm{i}kt} \cdot 1_{[-\pi,\pi]}(t) \cdot \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t && (10)\\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} c_k \cdot \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-\omega)t}\,\mathrm{d}t && (11)\\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} c_k \cdot \frac{\mathrm{e}^{\mathrm{i}(k-\omega)\pi} - \mathrm{e}^{-\mathrm{i}(k-\omega)\pi}}{\mathrm{i}(k-\omega)} && (12)\\
&= \sum_{k=-\infty}^{\infty} c_k \cdot \frac{\mathrm{e}^{\mathrm{i}(k-\omega)\pi} - \mathrm{e}^{-\mathrm{i}(k-\omega)\pi}}{\mathrm{i}2(k-\omega)\pi} && (13)\\
&= \sum_{k=-\infty}^{\infty} c_k \cdot \frac{\sin((k-\omega)\pi)}{(k-\omega)\pi} && (14)\\
&= \sum_{k=-\infty}^{\infty} c_k \cdot \frac{\sin(\omega\pi - k\pi)}{\omega\pi - k\pi} && (15)
\end{aligned}$$

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